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Retarded Differential Equations with Piecewise Constant Delays

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1. INTRODUCTION

Profound and close links exist between functional and functional differential equations. Thus, the study of the first often enables one to predict properties of differential equations of neutral type. On the other hand, some methods for the latter in the special case when the deviation of the argument vanishes at individual points have been used to investigate functional equations [1]. Functional equations are directly connected with difference equations of a discrete (for example, integer-valued) argument, the theory of which has been very intensively developed in the book [2] and in numerous subsequent papers. Bordering on difference equations are the impulse functional differential equations with impacts and switching, loaded equations (that is, those including values of the unknown solution for given constant values of the argument), equations

$$x'(t) = f(t, x(t), x(h(t)))$$

with lagging arguments of the form $h(t) = [t]$, that is, having intervals of constancy, etc. A substantial theory of such equations is virtually undeveloped [3].

In this paper we study differential equations with arguments $h(t) = [t]$, $h(t) = [t-n]$, and $h(t) = t - n[t]$. In the second section, linear equations with constant coefficients and two delays, $[t]$ and $[t-1]$, are considered. The initial-value problem is posed at $t = -1$ and $t = 0$, and the solution is sought for $t > 0$. The existence and uniqueness of solution and of its

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backward continuation on $(-\infty, 0]$ is proved. Furthermore, an important fact is established that the initial conditions may be posed at any two points $t_0 - 1$ and t_0, not necessarily integral. We show that equations with piecewise constant delays are closely related to impulse and loaded equations and, especially, to difference equations. In fact, the equations considered in this paper have the structure of continuous dynamical systems within intervals of unit length. Continuity of a solution at a point joining any two consecutive intervals then implies recursion relations for the values of the solution at such points. The equations are thus similar in structure to those found in certain "sequential-continuous" models of disease dynamics as treated by Busenberg and Cooke; see, in particular, the abstract formulation in Section III of [4]. The instrument of continued fractions plays an important role in the computation of solutions and in the study of their asymptotic behavior. Finally, necessary and sufficient conditions for asymptotic stability of the trivial solution are determined explicitly via coefficients of the given equation. In the third section the foregoing results are generalized for equations with many delays and systems of equations. Linear differential equations in Banach space with arguments $[t]$ and $t - n[t]$ are also considered. The properties of solutions to such equations with bounded operators are similar to those of solutions to systems of ordinary differential equations which can be viewed as equations in a finite-dimensional Banach space. In particular, it is not difficult to find the closed form for the solutions of linear equations with constant bounded operators. In the fourth section linear equations with variable operators are tackled. Some nonlinear equations are also investigated. Equations with unbounded delay [5] arise in cases of several argument deviations. In such problems the initial function is prescribed on $(-\infty, 0]$ and the solution is considered for $t > 0$.

2. Equations with Constant Coefficients

Consider the scalar initial-value problem

$$\begin{gathered} x'(t) = ax(t) + a_0 x([t]) + a_1 x([t-1]), \\ x(-1) = c_{-1}, \qquad x(0) = c_0 \end{gathered} \tag{2.1}$$

with constant coefficients. Here $[t]$ designates the greatest-integer function. This equation is very closely related to impulse and loaded equations. Indeed, write Eq. (2.1) as

$$x'(t) = ax(t) + \sum_{i=-\infty}^{\infty} (a_0 x(i) + a_1 x(i-1))(H(t-i) - H(t-i-1)),$$

where $H(t) = 1$ for $t > 0$ and $H(t) = 0$ for $t < 0$. If we admit distributional derivatives, then differentiating the latter relation gives

$$x''(t) = ax'(t) + \sum_{i=-\infty}^{\infty} (a_0 x(i) + a_1 x(i-1))(\delta(t-i) - \delta(t-i-1)),$$

where δ is the delta functional. This impulse equation contains the values of the unknown solution for the integral values of t. We introduce

DEFINITION 2.1. A solution of Eq. (2.1) on $[0, \infty)$ is a function $x(t)$ that satisfies the conditions:

(i) $x(t)$ is continuous on $[0, \infty)$.

(ii) The derivative $x'(t)$ exists at eact point $t \in [0, \infty)$, with the possible exception of the points $[t] \in [0, \infty)$ where one-sided derivatives exist.

(iii) Equation (2.1) is satisfied on each interval $[n, n+1) \subset [0, \infty)$ with integral endpoints.

Denote

$$b_0 = e^a + a^{-1} a_0 (e^a - 1), \qquad b_1 = a^{-1} a_1 (e^a - 1), \tag{2.2}$$

and let λ_1 and λ_2 be the roots of the equation

$$\lambda^2 - b_0 \lambda - b_1 = 0. \tag{2.3}$$

THEOREM 2.1. *Problem* (2.1) *has on* $[0, \infty)$ *a unique solution*

$$x(t) = c_{[t]} e^{a(t-[t])} + a^{-1}(a_0 c_{[t]} + a_1 c_{[t]-1})(e^{a(t-[t])} - 1), \tag{2.4}$$

where

$$c_{[t]} = (\lambda_1^{[t]+1}(c_0 - \lambda_2 c_{-1}) + (\lambda_1 c_{-1} - c_0)\lambda_2^{[t]+1})/(\lambda_1 - \lambda_2). \tag{2.5}$$

Proof. Assuming that $x_n(t)$ is a solution of Eq. (2.1) on the interval $n \leqslant t < +1$, with the conditions $x(n) = c_n$, $x(n-1) = c_{n-1}$, we have

$$x_n'(t) = ax(t) + a_0 c_n + a_1 c_{n-1}. \tag{2.6}$$

The general solution of this equation on the given interval is

$$x_n(t) = e^{a(t-n)} c - a^{-1}(a_0 c_n + a_1 c_{n-1}),$$

with an arbitrary constant c. Putting here $t = n$ gives

$$c_n = c - a^{-1}(a_0 c_n + a_1 c_{n-1}).$$

Hence,

$$c = (1 + a^{-1}a_0)\, c_n + a^{-1}a_1 c_{n-1}$$

and

$$x_n(t) = (a^{-1}a_1 c_{n-1} + (1 + a^{-1}a_0)\, c_n)\, e^{a(t-n)} - a^{-1}(a_0 c_n + a_1 c_{n-1}).$$

If $x_{n-1}(t)$ designates the solution of Eq. (2.1) on $[n-1, n)$ satisfying the conditions $x(n-1) = c_{n-1}$, $x(n-2) = c_{n-2}$, then

$$x_{n-1}(t) = (a^{-1}a_1 c_{n-2} + (1 + a^{-1}a_0)\, c_{n-1})\, e^{a(t-n+1)} - a^{-1}(a_0 c_{n-1} + a_1 c_{n-2}).$$

Since $x_{n-1}(n) = x_n(n)$, we obtain the recursion relation

$$c_n = (e^a + (e^a - 1)\, a^{-1}a_0)\, c_{n-1} + a^{-1}a_1(e^a - 1)\, c_{n-2}.$$

With the notations (2.2) it can be written as

$$c_n = b_0 c_{n-1} + b_1 c_{n-2}, \qquad n \geqslant 1. \tag{2.7}$$

We look for a particular solution of this difference equation in the form $c_n = \lambda^n$. Then

$$\lambda^n = b_0 \lambda^{n-1} + b_1 \lambda^{n-2},$$

and λ satisfies (2.3). If the roots λ_1 and λ_2 of (2.3) are different, the general solution of (2.7) is

$$c_n = k_1 \lambda_1^n + k_2 \lambda_2^n,$$

with arbitrary constants k_1 and k_2. In fact, it satisfies (2.7) for all integral n. In particular, for $n = -1$ and $n = 0$ this formula gives

$$\lambda_1^{-1} k_1 + \lambda_2^{-1} k_2 = c_{-1}, \qquad k_1 + k_2 = c_0$$

whence

$$k_1 = \lambda_1(c_0 - \lambda_2 c_{-1})/(\lambda_1 - \lambda_2), \qquad k_2 = \lambda_2(\lambda_1 c_{-1} - c_0)/(\lambda_1 - \lambda_2).$$

These results establish (2.5). If $\lambda_1 = \lambda_2 = \lambda$, then

$$c_n = \lambda^n (c_0(n+1) - \lambda c_{-1} n), \tag{2.8}$$

which is the limiting case of (2.5) as $\lambda_1 \to \lambda_2$. Formula (2.4) was obtained with the implicit assumption $a \neq 0$. If $a = 0$, then

$$x_n(t) = c_n + (a_0 c_n + a_1 c_{n-1})(t - n),$$

which is the limiting case of (2.4) as $a \to 0$. The uniqueness of the solution (2.4) on $[0, \infty)$ follows from its continuity and from the uniqueness of the problem $x_n(n) = c_n$ for (2.6) on each interval $[n, n+1]$.

COROLLARY. *The solution of* (2.1) *cannot grow to infinity faster than exponentially as* $t \to +\infty$.

Proof. Since $e^{a(t-[t])} \leqslant e^{|a|}$ we conclude from (2.4) that it remains to evaluate the coefficients $c_{[t]}$. From (2.5) we observe that if $\lambda_1 \neq \lambda_2$, then $|c_{[t]}| \leqslant km^t$, where k is some constant and $m = \max(|\lambda_1|, |\lambda_2|)$. And if $\lambda_1 = \lambda_2$, then from (2.8) it follows that $|c_{[t]}| \leqslant k(t+1)\, m^t$.

In ordinary differential equations with a continuous vector field the solution exists to the right and left of the initial t-value. For retarded functional differential equations (RFDE), this is not necessarily the case [6]. Since the solution of (2.1) on $[0, \infty)$ involves only the group e^{at}, it can be extended backwards on $(-\infty, 0]$.

THEOREM 2.2. *If* $a_1 \neq 0$, *the solution of* (2.1) *has a unique backward continuation on* $(-\infty, 0]$ *given by formulas* (2.4) *and* (2.5).

Proof. If $x_{-n}(t)$ denotes the solution of (2.1) on $[-n, -n+1)$ satisfying the conditions $x(-n) = c_{-n}$, $x(-n-1) = c_{-n-1}$, then from the equation

$$x'_{-n}(t) = ax_{-n}(t) + a_0 c_{-n} + a_1 c_{-n-1}$$

it follows that

$$x_{-n}(t) = e^{a(t+n)}c - a^{-1}(a_0 c_{-n} + a_1 c_{-n-1}),$$

where

$$c = (1 + a^{-1}a_0)\, c_{-n} + a^{-1}a_1 c_{-n-1}.$$

Therefore,

$$x_{-n}(t) = e^{a(t+n)}c_{-n} + a^{-1}(a_0 c_{-n} + a_1 c_{-n-1})(e^{a(t+n)} - 1). \tag{2.9}$$

This result proves (2.4) for $t < 0$. Since $x_{-n}(-n+1) = x_{-n+1}(-n+1) = c_{-n+1}$, we obtain the recursion relation

$$c_{-n+1} = b_0 c_{-n} + b_1 c_{-n-1}, \qquad n \geqslant 1, \tag{2.10}$$

with b_0 and b_1 from (2.2). The formula $c_{-n} = \lambda^{-n}$ gives Eq. (2.3) for λ and the general solution

$$c_{-n} = k_1 \lambda_1^{-n} + k_2 \lambda_2^{-n}$$

for (2.10). In particular,

$$\lambda_1^{-1}k_1 + \lambda_2^{-1}k_2 = c_{-1}, \qquad \lambda_1^{-2}k_1 + \lambda_2^{-2}k_2 = c_{-2},$$

and

$$k_1 = \lambda_1^2(c_{-1} - \lambda_2 c_{-2})/(\lambda_1 - \lambda_2), \qquad k_2 = \lambda_2^2(\lambda_1 c_{-2} - c_{-1})/(\lambda_1 - \lambda_2).$$

Since $a_1 \neq 0$, we have $b_1 \neq 0$, hence, we can find $c_{-2} = b_1^{-1}(c_0 - b_0 c_{-1})$ from (2.10) and substitute in the latter equations. Taking into account $b_0 = \lambda_1 + \lambda_2$ and $b_1 = -\lambda_1\lambda_2$ we obtain the formula

$$c_{-n} = (\lambda_1^{-n+1}(c_0 - \lambda_2 c_{-1}) + (\lambda_1 c_{-1} - c_0)\,\lambda_2^{-n+1})/(\lambda_1 - \lambda_2)$$

which together with (2.9) proves the theorem. If $a_1 = 0$, we formulate

THEOREM 2.3. *The problem*

$$x'(t) = ax(t) + a_0 x([t]), \qquad x(0) = c_0 \tag{2.11}$$

has on $[0, \infty)$ *a unique solution*

$$x(t) = u(t - [t])\,u^{[t]}(1)\,c_0, \tag{2.12}$$

where

$$u(t) = 1 + a^{-1}(e^{at} - 1)(a + a_0).$$

THEOREM 2.4. *If* $u(1) \neq 0$, *the solution of* (2.11) *has a unique backward continuation on* $(-\infty, 0]$ *given by formula* (2.12).

THEOREM 2.5. *If* $u(1) \neq 0$ *and* $u(t_0 - [t_0]) \neq 0$, *then Eq.* (2.11) *with the initial condition* $x(t_0) = x_0$ *has on* $(-\infty, \infty)$ *a unique solution*

$$x(t) = u(\{t\})\,u^{[t]-[t_0]}(1)\,u^{-1}(\{t_0\})\,x_0,$$

where $\{t\}$ *is the fractional part of* t.

The last theorem establishes an important fact that the initial-value problem for Eq. (2.11) may be posed at any point, not necessarily integral. A similar proposition is true also for Eq. (2.1).

THEOREM 2.6. *If* $a_1 \neq 0$ *and*

$$\lambda_i e^{a\{t_0\}} + a^{-1}(e^{a\{t_0\}} - 1)(\lambda_i a_0 + a_1) \neq 0, \qquad i = 1, 2 \tag{2.13}$$

where λ_i *are the roots of* (2.3), *then the problem* $x(t_0) = x_0$, $x(t_0 - 1) = x_{-1}$ *for Eq.* (2.1) *has a unique solution on* $(-\infty, \infty)$.

Proof. With the notations

$$m_0 = e^{a\{t_0\}} + a^{-1}a_0(e^{a\{t_0\}} - 1), \qquad n_0 = a^{-1}a_1(e^{a\{t_0\}} - 1),$$

and since $\{t_0 - 1\} = \{t_0\}$ we obtain from (2.4) the equations

$$m_0 c_{[t_0]} + n_0 c_{[t_0]-1} = x_0, m_0 c_{[t_0]-1} + n_0 c_{[t_0]-2} = x_{-1}. \tag{2.14}$$

Let

$$p_{[t_0]} = (\lambda_1^{[t_0]} - \lambda_2^{[t_0]})/(\lambda_1 - \lambda_2), \qquad q_{[t_0]} = (\lambda_1 \lambda_2^{[t_0]} - \lambda_2 \lambda_1^{[t_0]})/(\lambda_1 - \lambda_2).$$

Then from (2.5) we have

$$c_{[t_0]} = p_{[t_0]+1} c_0 + q_{[t_0]+1} c_{-1}, \qquad c_{[t_0]-1} = p_{[t_0]} c_0 + q_{[t_0]} c_{-1},$$
$$c_{[t_0]-2} = p_{[t_0]-1} c_0 + q_{[t_0]-1} c_{-1},$$

and substitute these expressions in (2.14):

$$(m_0 p_{[t_0]+1} + n_0 p_{[t_0]})\, c_0 + (m_0 q_{[t_0]+1} + n_0 q_{[t_0]})\, c_{-1} = x_0,$$
$$(m_0 p_{[t_0]} + n_0 p_{[t_0]-1})\, c_0 + (m_0 q_{[t_0]} + n_0 q_{[t_0]-1})\, c_{-1} = x_{-1}.$$

This system with the unknowns c_0 and c_{-1} has a unique solution if its determinant

$$\Delta = (\lambda_1 \lambda_2)^{[t_0]-1} (\lambda_1 \lambda_2 m_0^2 + (\lambda_1 + \lambda_2)\, m_0 n_0 + n_0^2)$$

is different from zero. The condition $a_1 \neq 0$ gives $\lambda_1 \lambda_2 \neq 0$, and $\Delta \neq 0$ iff $(\lambda_1 m_0 + n_0)(\lambda_2 m_0 + n_0) \neq 0$ which is equivalent to (2.13).

The theory of continued fractions [7] also provides a useful instrument in the study of (2.1). In this manner, we can easily compute the coefficients c_n. Let $r_n = c_n/c_{n-1}$, then from (2.7)

$$r_1 = b_0 + b_1(c_{-1}/c_0).$$

After this we find

$$r_2 = b_0 + b_1/r_1 = b_0 + \cfrac{b_1}{b_0 + \cfrac{b_1}{c_0/c_{-1}}},$$

$$r_3 = b_0 + b_1/r_2 = b_0 + \cfrac{b_1}{b_0 + \cfrac{b_1}{b_0 + \cfrac{b_1}{c_0/c_{-1}}}}.$$

This procedure leads to the representation of r_n by a finite continued fraction

$$r_n = b_0 + \frac{b_1}{b_0+} \frac{b_1}{b_0+} \cdots \frac{b_1}{b_0+} \frac{b_1}{c_0/c_{-1}}.$$

Then

$$c_n = r_1 r_2 \cdots r_n c_0.$$

As $t \to \infty$, the continued fraction for $r_{[t]}$ becomes infinite,

$$b_0 + \frac{b_1}{b_0+} \frac{b_1}{b_0+} \cdots \frac{b_1}{b_0+} \cdots \tag{2.15}$$

and questions arise about its convergence. It is well known [7] that the continued fraction

$$a_0 + \frac{1}{a_1+} \frac{1}{a_2+} \frac{1}{a_3+} \cdots = [a_0; a_1, a_2, a_3, \ldots]$$

with positive components a_n $(n \geqslant 1)$ converges iff the series $\sum a_n$ diverges. The fraction (2.15) can be changed to $[b_0; b_0/b_1, b_0, b_0/b_1, b_0, \cdots]$. Therefore, if $b_0 > 0$, $b_1 > 0$, then (2.15) converges. If $b_0 < 0$, $b_1 > 0$, then (2.15) also converges since it can be transformed into $-[|b_0|; |b_0|/b_1, |b_0|, |b_0|/b_1, \ldots]$. In the case $b_0 > 0$, $b_1 < 0$ we employ a theorem [8] which states that the continued fraction

$$b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_2+} \cdots \frac{a_i}{b_i+} \cdots$$

with positive terms b_i $(i \geqslant 1)$ converges if $b_i - |a_i| > 1$, for all $i \geqslant 1$. Regarding (2.15) it means that this fraction converges if

$$b_0 + b_1 > 1.$$

For $b_0 < 0$, $b_1 < 0$ we change (2.15) to

$$-\left(|b_0| + \frac{b_1}{|b_0|+} \frac{b_1}{|b_0|+} \cdots \frac{b_1}{|b_0|+} \cdots\right)$$

and conclude that it converges if

$$-b_0 + b_1 > 1.$$

Combining the last two inequalities gives a sufficient condition

$$|b_0| + b_1 > 1$$

for the convergence of (2.15) when $b_1 < 0$. From here it follows that

$$b_0^2 + 4b_1 > (1 + b_1)^2. \tag{2.16}$$

Thus, we proved

THEOREM 2.7. *The continued fraction* (2.15) *and* c_n/c_{n-1} *converge to the same limit if either of the following hypotheses is satisfied:*

(i) $b_1 > 0$, $b_0 \neq 0$;

(ii) $b_1 < 0$, $|b_0| + b_1 > 1$.

THEOREM 2.8. *Under the conditions of Theorem* 2.7, *the continued fraction* (2.15) *converges to the root of Eq.* (2.3) *which has the greater absolute value.*

Proof. Inequality (2.16) guarantees that the roots of (2.3) are real and have distinct absolute values. We derived (2.15) from the difference equation (2.7) as a result of the limiting process in the finite continued fraction for c_n/c_{n-1}. On the other hand, (2.15) arises also as a formal expansion in a continued fraction of a root of (2.3). We have

$$\lambda = \frac{\lambda^2}{\lambda} = \frac{b_0\lambda + b_1}{\lambda} = b_0 + \frac{b_1}{\lambda} = b_0 + \cfrac{b_1}{b_0 + \cfrac{b_1}{\lambda}}.$$

Continuing this procedure leads to

$$\lambda = b_0 + \frac{b_1}{b_0+}\,\frac{b_1}{b_0+}\cdots\frac{b_1}{b_0+}\cdots. \tag{2.17}$$

If $b_1 > 0$, $b_0 > 0$, this fraction converges to the positive root of (2.3) which is greater than the absolute value of the other (negative) root. For $b_1 > 0$, $b_0 < 0$ we change (2.17) to

$$\lambda = -\left(|b_0| + \frac{b_1}{|b_0|+}\,\frac{b_1}{|b_0|+}\cdots\frac{b_1}{|b_0|+}\cdots\right).$$

In this case (2.15) converges to the negative root of (2.3) the absolute value of which is greater than the other (positive) root. Let $D = b_0^2/4 + b_1$ where $b_1 < 0$, and let

$$D^{1/2} = \frac{|b_0|}{2} - \frac{1}{y}, \qquad y > 0.$$

Then

$$y = -\left(\frac{|b_0|}{2} + D^{1/2}\right)\Big/ b_1$$

and

$$D^{1/2} = \frac{|b_0|}{2} + \frac{b_1}{\dfrac{|b_0|}{2} + D^{1/2}}.$$

From here we obtain

$$D^{1/2} = \frac{|b_0|}{2} + \frac{b_1}{|b_0|+} \frac{b_1}{|b_0|+} \cdots \frac{b_1}{|b_0|+} \cdots$$

and the formula

$$\lambda_{1,2} = \frac{b_0}{2} \pm \left(\frac{|b_0|}{2} + \frac{b_1}{|b_0|+} \cdots \frac{b_1}{|b_0|+} \cdots\right)$$

for the roots of (2.3). If $b_0 > 0$, the root of greater absolute value is $\lambda_1 = b_0/2 + D^{1/2}$, where

$$D^{1/2} = \frac{b_0}{2} + \frac{b_1}{b_0+} \frac{b_1}{b_0+} \cdots \frac{b_1}{b_0+} \cdots,$$

and λ_1 is the limit of continued fraction (2.15). When $b_0 < 0$, the root of greater absolute value is $\lambda_2 = b_0/2 - D^{1/2}$, where

$$D^{1/2} = -\frac{b_0}{2} + \frac{b_1}{-b_0+} \frac{b_1}{-b_0+} \cdots \frac{b_1}{-b_0+} \cdots = -\left(\frac{b_0}{2} + \frac{b_1}{b_0+} \cdots\right),$$

and λ_2 is the limit of (2.15).

THEOREM 2.9. *The solution $x = 0$ of Eq.* (2.1) *is asymptotically stable as $t \to +\infty$ iff the moduli of the roots of Eq.* (2.3) *satisfy the inequalities*

$$|\lambda_1| < 1, \qquad |\lambda_2| < 1. \tag{2.18}$$

Proof. It follows from (2.4) and (2.5) since $e^{a(t)}$ is bounded and $c_{[t]} \to 0$ as $t \to +\infty$ iff (2.18) holds.

It is easy to see that if (2.18) and the hypotheses of Theorem 2.7 are fulfilled, the series $\sum c_n$ converges absolutely. By resorting to formula (2.5) we can relax the conditions of this proposition.

THEOREM 2.10. *The series* $\sum |c_n|$ *converges if the roots of* (2.3) *satisfy* (2.18).

THEOREM 2.11. *If the solution* $x = 0$ *of Eq.* (2.1) *is asymptotically stable as* $t \to +\infty$, *then*

$$\begin{gathered} -a(2+e^a)/(e^a-1) < a_0 < a(2-e^a)/(e^a-1), \\ |a_1| < a/(e^a-1). \end{gathered} \tag{2.19}$$

Proof. From (2.18) it follows that

$$|b_1| = |\lambda_1 \lambda_2| < 1, \qquad |b_0| = |\lambda_1 + \lambda_2| < 2,$$

which together with (2.2) gives (2.19).

THEOREM 2.12. *The solution* $x = 0$ *of Eq.* (2.1) *is asymptotically stable as* $t \to +\infty$ *iff any one of the following hypotheses is satisfied:*

(i)
$$-a(2+e^a)/(e^a-1) < a_0 < -ae^a/(e^a-1),$$
$$-\frac{a(e^a + a^{-1}(e^a-1)\,a_0)^2}{4(e^a-1)} \leqslant a_1 < \frac{a(e^a+1)}{e^a-1} + a_0;$$

(ii)
$$-\frac{ae^a}{e^a-1} < a_0 < \frac{a(2-e^a)}{e^a-1},$$
$$-\frac{a(e^a + a^{-1}(e^a-1)\,a_0)^2}{4(e^a-1)} \leqslant a_1 < -a - a_0;$$

(iii)
$$-\frac{a(2+e^a)}{e^a-1} < a_0 < \frac{a(2-e^a)}{e^a-1},$$
$$-\frac{a}{e^a-1} < a_1 < -\frac{a(e^a + a^{-1}(e^a-1)\,a_0)^2}{4(e^a-1)}.$$

Proof. According to (2.18), the necessary and sufficient condition for asymptotic stability of (2.1) takes the form

$$|b_0 \pm \sqrt{b_0^2 + 4b_1}| < 2.$$

First, assume that the roots of (2.3) are real, then

$$b_0 + \sqrt{b_0^2 + 4b_1} < 2, \qquad b_0 - \sqrt{b_0^2 + 4b_1} > -2.$$

From here we obtain the following system of inequalities:

$$-2 < b_0 < 2, \qquad -1 < b_1 < 1, \qquad b_1 < 1 + b_0,$$
$$b_1 < 1 - b_0, \qquad b_1 \geqslant -b_0^2/4.$$

If $-2 < b_0 < 0$, then $-b_0^2/4 \leqslant b_1 < b_0 + 1$. Hence, if $-2 < e^a + a^{-1}(e^a - 1)\, a_0 < 0$, then

$$-(e^a + a^{-1}(e^a - 1)\, a_0)^2/4 \leqslant a^{-1}(e^a - 1)\, a_1 < e^a + 1 + a^{-1}(e^a - 1)\, a_0,$$

which is equivalent to (i). In the case $0 < b_0 < 2$ we have $-b_0^2/4 \leqslant b_1 < 1 - b_0$, that is, $0 < e^a + a^{-1}(e^a - 1)\, a_0 < 2$ is combined with

$$-(e^a + a^{-1}(e^a - 1)\, a_0)^2/4 \leqslant a^{-1}(e^a - 1)\, a_1 < 1 - e^a - a^{-1}(e^a - 1)\, a_0.$$

This gives (ii). It remains to consider the case when the roots of (2.3) are complex. Then

$$b_0^2 + 4b_1 < 0, \qquad |b_0 \pm i\sqrt{-b_0^2 - 4b_1}| < 2.$$

From here, $-1 < b_1 < -b_0^2/4$, and with the notations (2.2),

$$-1 < a^{-1}(e^a - 1)\, a_1 < -a(e^a + a^{-1}(e^a - 1)\, a_0)^2/4(e^a - 1).$$

To obtain (iii), we associate with these inequalities the first of conditions (2.19).

3. Some Generalizations

Let $x_n(t)$ be a solution of the equation

$$x'(t) = ax(t) + \sum_{i=0}^{N} a_i x([t - i]), \qquad a_N \neq 0, \tag{3.1}$$

with constant coefficients on the interval $n \leqslant t < n + 1$. If we set $x(n - i) = c_{n-i}$, $0 \leqslant i \leqslant N$, then we have the equation

$$x_n'(t) = ax_n(t) + \sum_{i=0}^{N} a_i c_{n-i},$$

the general solution of which is

$$x_n(t) = e^{a(t-n)}c - \sum_{i=0}^{N} a^{-1} a_i c_{n-i}, \qquad a \neq 0.$$

For $t = n$ this gives

$$c_n = c - \sum_{i=0}^{N} a^{-1} a_i c_{n-i}$$

and

$$x_n(t) = e^{a(t-n)} c_n + (e^{a(t-n)} - 1) \sum_{i=0}^{N} a^{-1} a_i c_{n-i}. \tag{3.2}$$

Changing here n to $n-1$ and taking into account that $x_{n-1}(n) = x_n(n) = c_n$ we obtain

$$c_n = (e^a + (e^a - 1) a^{-1} a_0) c_{n-1} + (e^a - 1) a^{-1} \sum_{i=1}^{N} a_i c_{n-i-1}.$$

With the notations

$$b_0 = e^a + (e^a - 1) a^{-1} a_0, \qquad b_i = (e^a - 1) a^{-1} a_i, \qquad i \geqslant 1$$

this equation takes the form

$$c_n - b_0 c_{n-1} - b_1 c_{n-2} - \cdots - b_N c_{n-N-1} = 0. \tag{3.3}$$

Its particular solution is sought as $c_n = \lambda^n$; then

$$\lambda^{N+1} - b_0 \lambda^N - b_1 \lambda^{N-1} - \cdots - b_N = 0. \tag{3.4}$$

If all roots $\lambda_1, \lambda_2, \ldots, \lambda_{N+1}$ of (3.4) are simple, the general solution of (3.3) is given by

$$c_n = k_1 \lambda_1^n + k_2 \lambda_2^n + \cdots + k_{N+1} \lambda_{N+1}^n, \tag{3.5}$$

with arbitrary constant coefficients. The initial-value problem for (3.1) may be posed at any $N+1$ consecutive points. Thus we consider the existence and uniqueness of the solution to (3.1) for $t > N$ satisfying the conditions

$$x(i) = c_i, \qquad i = 0, 1, \ldots, N. \tag{3.6}$$

Then letting $n = 0, 1, \ldots, N$ and $c_n = x(n)$ in (3.5) we get a system of equations with Vandermond's determinant $\det(\lambda_j^i)$ which is different from zero. Hence, the unknowns k_j are uniquely determined by (3.6). If some roots of (3.4) are multiple, the general solution of (3.3) contains products of exponential functions by polynomials of certain degree. The limiting case of (3.2) as $a \to 0$ gives the solution of (3.1) when $a = 0$. We proved

THEOREM 3.1. *Problem* (3.1)–(3.6) *has a unique solution on* $[N, \infty)$. *This solution cannot grow to infinity faster than exponentially as* $t \to +\infty$.

THEOREM 3.2. *The solution* $x = 0$ *of* (3.1) *is asymptotically stable as* $t \to +\infty$ *iff*

$$|\lambda_j| < 1, \qquad j = 1, 2, ..., N + 1.$$

For the problem

$$\begin{gathered} x'(t) = Ax(t) + A_0 x([t]) + A_1 x([t-1]), \\ x(-1) = c_{-1}, \qquad x(0) = c_0 \end{gathered} \tag{3.7}$$

in which A, A_0, A_1 are $r \times r$-matrices and x is an r-vector, let $x_n(t)$ designate the solution of Eq. (3.7) on $[n, n+1)$ satisfying the conditions $x(n-1) = c_{n-1}$, $x(n) = c_n$. Then, if A is nonsingular,

$$x_n(t) = (e^{A(t-n)} + (e^{A(t-n)} - I) A^{-1} A_0) c_n + (e^{A(t-n)} - I) A^{-1} A_1 c_{n-1}. \tag{3.8}$$

Changing n to $n-1$ and taking into account $x_{n-1}(n) = x_n(n)$ we obtain the recursion relation

$$c_n = B_0 c_{n-1} + B_1 c_{n-2}, \qquad n \geqslant 1, \tag{3.9}$$

where

$$B_0 = e^A + (e^A - I) A^{-1} A_0, \qquad B_1 = (e^A - I) A^{-1} A_1.$$

Looking for a nonzero solution $c_n = \lambda^n k$, with a constant vector k, we conclude that λ satisfies the equation

$$\det(\lambda^2 I - \lambda B_0 - B_1) = 0 \tag{3.10}$$

which has $2r$ nontrivial solutions if $\det B_1 \neq 0$. Assuming that these roots are simple we write the general solution of (3.9)

$$c_n = \lambda_1^n k_1 + \lambda_2^n k_2 + \cdots + \lambda_{2r}^n k_{2r}$$

with constant vectors k_j each of which depends on the corresponding value λ_j and contains one arbitrary scalar factor. These factors can be found from the initial conditions (3.7). If some of the λ_j are multiple zeros of (3.10), then the expression for c_n includes also products of exponentials by polynomials of n.

THEOREM 3.3. *Assume that in Eq.* (3.7) *the matrices* A, $e^A - I$, *and* A_1 *are nonsingular. Then problem* (3.7) *has a unique solution on* $[0, \infty)$, *and this solution cannot grow to infinity faster than exponentially.*

THEOREM 3.4. *The solution* $x = 0$ *of Eq.* (3.7) *is asymptotically stable as* $t \to +\infty$ *if all zeros of* (3.10) *are located in the open unit disk.*

We note that the unknowns c_n in (3.3) can be computed also by means of the so-called branching continued fractions [9], without resorting to the characteristic equation (3.4). The instrument of matrix continued fractions [10] may be used to find the vectors c_n in (3.9). Equation (3.10) is often encountered in various problems of quantum mechanics and theory of small oscillations. Some sufficient conditions for all zeros of (3.10) to lie in the open unit disk may be found in [11]. Let

$$B_0 = (b_{ij}^{(0)}), \qquad B_1 = (b_{ij}^{(1)}), \qquad i, j = 1, 2, ..., r.$$

Then $|\lambda| < 1$ for all zeros of (3.10) if

$$\sum_{j=1}^{r} (|b_{ij}^{(0)}| + |b_{ij}^{(1)}|) < 1, \qquad i = 1, 2, ..., r.$$

Consider in a Banach space E the equation

$$x'(t) = Ax(t) + Bx([t]) \tag{3.11}$$

with linear constant operators $A : D(A) \to E$ and $B : D(B) \to E$, their domains $D(A) \subset D(B) \subset E$, and $D(A)$ is everywhere dense in E. We introduce the definition of a solution to (3.11) that modifies Definition 2.1 and the given in [12] for the equation

$$x' = Ax. \tag{3.12}$$

DEFINITION 3.1. A solution of Eq. (3.11) on $[0, \infty)$ is a function $x(t)$ satisfying the conditions:

(i) $x(t)$ is continuous on $[0, \infty)$ and its values lie in the domain $D(A)$ for all $t \in [0, \infty)$.

(ii) At each point $t \in [0, \infty)$ there exists a strong derivative $x'(t)$, with the possible exception of the points $[t] \in [0, \infty)$ where one-sided derivatives exist.

(iii) Equation (2.1) is satisfied on each interval $[n, n+1) \subset [0, \infty)$ with integral endpoints.

The Cauchy problem on $[0, \infty)$ is to find a solution of the equation on $[0, \infty)$ satisfying the initial condition

$$x(0) = c_0 \in D(A). \tag{3.13}$$

Let $L(E, E)$ be the space of bounded linear operators from E into E. It is easy to establish the existence and uniqueness of solution to problem (3.11)–(3.13) as well as its exponential growth and backward continuation.

THEOREM 3.5. *If A, $B \in L(E, E)$ and A is bijective, then problem* (3.11)–(3.13) *on* $[0, \infty)$ *has a unique solution*

$$x(t) = V(t - [t])\, V^{[t]}(1)\, c_0, \tag{3.14}$$

where

$$V(t) = e^{At} + (e^{At} - I) A^{-1} B.$$

This solution cannot grow to infinity faster than exponentially. If, in addition, there exists a bounded inverse of the operator $V(1)$, then the solution of (3.11)–(3.13) has a unique backward continuation on $(-\infty, 0]$ given by formula (3.14).

Remark. It is well known that if the spectrum of the operator $A \in L(E, E)$ lies in the open left half-plane, then for any bounded function $f(t) \in E$:

$$\sup_{0 \leqslant t < \infty} \|f(t)\| < \infty,$$

all solutions of the nonhomogeneous equation

$$x'(t) = Ax(t) + f(t)$$

are bounded on $[0, \infty)$. In general, this is not true for (3.11) which is a nonhomogeneous equation with a constant free term on each interval $[n, n+1)$. For instance, the solution

$$x(t) = (2 - e^{-\{t\}})(2 - e^{-1})^{[t]}$$

of the scalar problem

$$x'(t) = -x(t) + 2x([t]), \qquad x(0) = 1$$

is unbounded on $[0, \infty)$ since

$$|x(t)| \geqslant (2 - e^{-1})^{[t]}.$$

The reason for this is the change of the free term as t passes through integral values.

THEOREM 3.6. *If $A, B \in L(E, E)$, then the equation*

$$x'(t) = Ax(t) + Bx(t - [t]) \tag{3.15}$$

with the initial condition (3.13) *has a unique solution on* $[0, \infty)$:

$$x(t) \equiv x_0(t) = e^{(A+B)t}c_0, \qquad 0 \leqslant t \leqslant 1,$$

$$\begin{aligned} x(t) = e^{At}c_0 + \sum_{k=1}^{[t]} \int_{k-1}^{k} e^{A(t-s)}Bx_0(s-k+1)\,ds \\ + \int_{[t]}^{t} e^{A(t-s)}Bx_0(s-[t])\,ds, \qquad t \geqslant 1. \end{aligned} \tag{3.16}$$

Proof. Relation (3.15) represents an interesting example of an equation with infinite delay $[t]$ that admits a pointwise initial condition. Let $x_n(t)$ be the solution of (3.15) on $[n, n+1)$ satisfying $x(n) = c_n$. Then

$$x_n'(t) = Ax_n(t) + Bx_n(t-n), \qquad n \geqslant 0,$$

and for $x_0(t)$ this gives the first part of (3.16). For $n \geqslant 1$ we have $0 \leqslant t - n < 1$, hence

$$x_n'(t) = Ax_n(t) + Bx_0(t-n), \qquad x_n(n) = c_n.$$

From here

$$x_n(t) = e^{A(t-n)}c_n + \int_n^t e^{A(t-s)}Bx_0(s-n)\,ds \tag{3.17}$$

and

$$x_{n-1}(t) = e^{A(t-n+1)}c_{n-1} + \int_{n-1}^{t} e^{A(t-s)}Bx_0(s-n+1)\,ds.$$

Since $x_{n-1}(n) = x_n(n)$, we put $t = n$ to get

$$c_n = e^{A}c_{n-1} + \int_{n-1}^{n} e^{A(n-s)}Bx_0(s-n+1)\,ds.$$

Applying this formula successively to (3.17) yields (3.16).

THEOREM 3.7. *In the conditions of Theorem* 3.6 *solution* (3.16) *has a unique backward continuation on* $(-\infty, 0]$:

$$\begin{aligned} x(t) = e^{At}c_0 - \sum_{k=-1}^{[t]} \int_{k}^{k+1} e^{A(t-s)}Bx_0(s-k)\,ds \\ + \int_{[t]}^{t} e^{A(t-s)}Bx_0(s-[t])\,ds. \end{aligned} \tag{3.18}$$

Proof. For the solution $x_{-n}(t)$ of (3.15) on $[-n, -n+1)$ we have the equation

$$x'_{-n}(t) = Ax_{-n}(t) + Bx_0(t+n),$$

for which we put $x_{-n}(-n) = c_{-n}$. Then

$$x_{-n}(t) = e^{A(t+n)}c_{-n} + \int_{-n}^{t} e^{A(t-s)}Bx_0(s+n)\,ds.$$

Similarly, on $[-n+1, -n+2)$ this gives

$$x_{-n+1}(t) = e^{A(t+n-1)}c_{-n+1} + \int_{-n+1}^{t} e^{A(t-s)}Bx_0(s+n-1)\,ds.$$

Since $x_{-n}(-n+1) = x_{-n+1}(-n+1) = c_{-n+1}$, it follows that

$$c_{-n+1} = e^{A}c_{-n} + \int_{-n}^{-n+1} e^{A(-n+1-s)}Bx_0(s+n)\,ds,$$

whence

$$c_{-n} = e^{-A}c_{-n+1} - \int_{-n}^{-n+1} e^{A(-n-s)}Bx_0(s+n)\,ds.$$

This formula leads to (3.18).

The Cauchy problem (3.12)–(3.13) is correctly posed on $[0, \infty)$ if for any $c_0 \in D(A)$ it has a unique solution, and this solution depends continuously on the initial data in the sense that if $x_n(0) \to 0 (x_n(0) \in D(A))$, then $x_n(t) \to 0$ for the corresponding solution at every $t \in [0, \infty)$. If the Cauchy problem for (3.12) is correct, its solution is given by the formula

$$x(t) = U(t)\,c_0 \qquad (c_0 \in D(A)),$$

where $U(t)$ is a semigroup of operators strongly continuous for $t > 0$. For many applications it is necessary to extend the concept of solution of the Cauchy problem. A weakened solution of (3.12) on $[0, \infty)$ is a function $x(t)$ which is continuous on $[0, \infty)$, strongly continuously differentiable on $(0, \infty)$ and satisfies the equation there. By a weakened Cauchy problem on $[0, \infty)$ we mean the problem of finding a weakened solution satisfying the initial condition $x(0) = c_0$. Here the element c_0 may already not lie in the domain of the operator A. Thus, the demands on the behavior of the solution at zero are relaxed. On the other hand, we require the continuity of the derivative of the solution for $t > 0$. However, for a correct Cauchy problem this requirement is automatically satisfied [12].

THEOREM 3.8. *Suppose that Eq.* (3.11) *with linear constant operators A and B satisfies the following hypotheses.*

(i) *The operator A is closed and has at least one regular point, the domain $D(A)$ is dense in E.*

(ii) *The weakened Cauchy problem for* (3.12) *is correct on $D(A)$.*

(iii) $D(B) \supset D(A)$ *and* $Bx \in D(A)$, *for any* $x \in D(A)$.

Then on $[0, \infty)$ *problem* (3.11)–(3.13) *has a unique solution*

$$x(t) = \left(U(t - [t]) + \int_{[t]}^{t} U(t-s)\,B\,ds\right) \times \prod_{k=[t]}^{1} \left(U(1) + \int_{k-1}^{k} U(k-s)\,B\,ds\right) c_0. \tag{3.19}$$

Proof. If $x_n(t)$ is a weakened solution of (3.11) on $[n, n+1)$, then, by virtue of (i) and (ii), it can be represented in the form

$$x_n(t) = U(t-n)\,c_n + \int_n^t U(t-s)\,Bc_n\,ds, \tag{3.20}$$

where $c_n = x(n)$. If $c_n \in D(A)$, the first term in the right of (3.20) is a solution of (3.12). Since the term Bc_n is constant, the integral in (3.20) really yields a weakened solution of (3.11). Furthermore, from (iii) we have $Bc_n \in D(A)$. Hence, this integral gives a particular solution of (3.11), and the assumption $c_n \in D(A)$ enables us to maintain that (3.20) is a solution of (3.11) in the given interval. If $c_{n-1} \in D(A)$, then

$$x_{n-1}(t) = U(t-n+1)\,c_{n-1} + \int_{n-1}^{t} U(t-s)\,Bc_{n-1}\,ds,$$

$$(x_{n-1}(n-1) = c_{n-1})$$

is a solution of (3.11) for $n-1 \leqslant t < n$. The recursion relation

$$c_n = \left(U(1) + \int_{n-1}^{n} U(n-s)\,B\,ds\right) c_{n-1}$$

which follows from the requirement $x_{n-1}(n) = x_n(n)$ leads to (3.19). Since $c_0 \in D(A)$ and $x_0(t)$ is the solution of (3.11) on $[0, 1]$, then $c_1 = x_0(1) \in D(A)$. Now we consider the solution $x_1(t)$ on $[1, 2]$ and see that $c_2 = x_1(2) \in D(A)$. Continuing the solution to the right we conclude that $c_n \in D(A)$ for any n.

If the correctness of the Cauchy problem is not known beforehand, we may resort to certain restrictions on the resolvent of the operator A in order to prove the existence and uniqueness of the solution. There are different conditions of this kind. We follow the approach suggested in [13]. Let α be any real number, $\theta \in (0, \pi/2)$, and let $G = G(\alpha, \theta)$ be the domain that contains the semiaxis $(\alpha, +\infty)$ and is bounded by two rays from the point $(\alpha, 0)$ forming angles θ and $-\theta$ with the negative direction of the real axis. The operator A is called an abstract elliptic operator if there exist a constant $\mu > 0$ and a domain G (of the indicated type) such that the resolvent set of A contains G and for all $\lambda \in G$

$$\|(A - \lambda I)^{-1}\| \leqslant \frac{\mu}{1 + |\lambda|}.$$

We state the following:

THEOREM 3.9. *Problem* (3.11)–(3.13) *has a unique solution on* $[0, \infty)$ *given by formula* (3.19) *if A is a closed abstract elliptic operator with the domain $D(A)$ dense in E, and the operator $B : D(B) \to D(A)$ where $D(B) \supset D(A)$.*

Now we consider in a Banach space E the equation

$$x'(t) = \sum_{i=0}^{N} A_i x(t - i[t]) \tag{3.21}$$

with linear constant operators $A_i : D(A) \to E$, having the same domain $D(A)$ dense in E.

DEFINITION 3.2. The function $x(t)$ is called a solution of the initial-value problem for (3.21), if the following conditions are satisfied:

(i) The function $x(t)$ is continuous on $(-\infty, \infty)$ and its values lie in the domain $D(A)$ for all $t \in (-\infty, \infty)$.

(ii) At each point $t \in [0, \infty)$ there exists a strong derivative $x'(t)$, with the possible exception of the points $[t]$ where one-sided derivatives exist.

(iii) On $(-\infty, 0]$, $x(t)$ coincides with a given continuous function $\phi(t) \in D(A)$ and satisfies Eq. (3.21) on each interval $[n, n+1)$.

The solution $x_0(t)$ of (3.21) on $[0, 1)$ satisfies the equation

$$x_0'(t) = \left(\sum_{i=0}^{N} A_i\right) x_0(t), \qquad x_0(0) = \phi(0). \tag{3.22}$$

We also employ the homogeneous equation

$$x'(t) = A_0 x(t) \tag{3.23}$$

corresponding to (3.21) for $t \geqslant 1$. Denote

$$f_k(t) = A_1 x_0(t-k) + \sum_{i=2}^{N} A_i \phi(t-ik), \qquad k = 1, 2, \ldots \tag{3.24}$$

$$(k \leqslant t \leqslant k+1).$$

THEOREM 3.10. *Suppose that Eq.* (3.21) *satisfies the following hypotheses:*

(i) *The Cauchy problem for* (3.22) *is correct.*

(ii) *The Cauchy problem for* (3.23) *is uniformly correct.*

(iii) *The values* $f_i(t) \in D(A)$, *the functions* $f_i(t)$ *and* $A_0 f_i(t)$ *are continuous.*

Then the initial-value problem for (3.21) *has a unique solution*

$$\begin{aligned} x(t) = U(t-1)\, x_0(1) &+ \sum_{k=1}^{[t-1]} \int_k^{k+1} U(t-s) f_k(s)\, ds \\ &+ \int_{[t]}^{t} U(t-s) f_{[t]}(s)\, ds, \qquad t \geqslant 1, \end{aligned} \tag{3.25}$$

where $U(t)$ *is the semigroup operator generated by* (3.23).

Proof. For $t \in [n, n+1)$, Eq. (3.21) takes the form

$$x'(t) = A_0 x(t) + A_1 x_0(t-n) + \sum_{i=2}^{N} A_i \phi(t-in),$$

which also can be written

$$x'(t) = A_0 x(t) + f_n(t). \tag{3.26}$$

By virtue of (ii) and (iii) and Theorem 6.5 [12], the formula

$$x_n(t) = U(t-n)\, c_n + \int_n^t U(t-s) f_n(s)\, ds \tag{3.27}$$

yields the solution of (3.26) with the condition $x_n(n) = c_n$. On $n-1 \leqslant t < n$ we have

$$x_{n-1}(t) = U(t-n+1)\, c_{n-1} + \int_{n-1}^{t} U(t-s) f_{n-1}(s)\, ds,$$

where $x_{n-1}(n-1) = c_{n-1}$. Therefore, the relation $x_{n-1}(n) = c_n$ gives

$$c_n = U(1)\, c_{n-1} + \int_{n-1}^{n} U(n-s) f_{n-1}(s)\, ds.$$

From here

$$c_n = U(1) \left(U(1)\, c_{n-2} + \int_{n-2}^{n-1} U(n-1-s) f_{n-2}(s)\, ds \right)$$
$$+ \int_{n-1}^{n} U(n-s) f_{n-1}(s)\, ds.$$

The property

$$U(t_1)\, U(t_2) = U(t_1 + t_2), \qquad 0 \leqslant t_1, \quad t_2 < \infty,$$

implies

$$c_n = U(2)\, c_{n-2} + \int_{n-2}^{n-1} U(n-s) f_{n-2}(s)\, ds$$
$$+ \int_{n-1}^{n} U(n-s) f_{n-1}(s)\, ds.$$

Continuing this procedure leads to the formula

$$c_n = U(n-1)\, c_1 + \sum_{k=1}^{n-1} \int_{k}^{k+1} U(n-s) f_k(s)\, ds, \qquad n \geqslant 1,$$

which together with (3.27) proves the theorem. It is clear that $c_1 = x_0(1) \in D(A)$ because $x_0(t)$ is the solution of (3.22) where $\phi(0) \in D(A)$.

THEOREM 3.11. *If the operators $A_i \in L(E, E)$, the initial-value problem for* (3.21) *has a unique solution*

$$x(t) = e^{A_0(t-1)} e^{A} \phi(0) + \sum_{k=1}^{[t-1]} \int_{k}^{k+1} e^{A_0(t-s)} f_k(s)\, ds$$
$$+ \int_{[t]}^{t} e^{A_0(t-s)} f_{[t]}(s)\, ds, \qquad t \geqslant 1, \tag{3.28}$$

where $A = \sum_{i=0}^{N} A_i$ and the functions $f_k(s)$ are defined by (3.24) *with $x_0(s) = e^{As}\phi(0)$.*

It is well known [6] that the solution of the scalar differential difference equation

$$x'(t) = Ax(t) + Bx(t-r),$$
$$x(t) = \phi(t), \qquad t \in [-r, 0],$$

satisfies the exponential estimate

$$|x(t)| \leqslant a\,|\phi|\,e^{bt}, \qquad t \geqslant 0$$

with positive constants a, b and $|\phi| = \sup |\phi(t)|$ for $-r \leqslant t \leqslant 0$. Similar results hold also for (3.21).

THEOREM 3.12. *If in the conditions of Theorem* 3.10 *the functions* $f_k(t)$ *are bounded in* $D(A)$ *uniformly relative to* t *and* k*:*

$$\|f_k(t)\| \leqslant C, \qquad (k \leqslant t \leqslant k+1), \qquad k = 1, 2, \ldots,$$

then solution (3.25) *of the initial-value problem for Eq.* (3.21) *satisfies the inequality*

$$\|x(t)\| \leqslant Le^{wt}, \qquad t \geqslant 0, \tag{3.29}$$

where L *and* w *are some positive constants.*

Proof. The estimate

$$\|U(t)\| \leqslant Me^{wt}, \qquad t \geqslant 0, \quad M, w > 0,$$

holds for the semigroup $U(t)$ generated by a uniformly correct Cauchy problem. Therefore, it follows from (3.25) that

$$\|x(t)\| \leqslant M\,\|x_0(1)\|\,e^{-w}e^{wt} + CM \sum_{k=1}^{[t]-1} \int_k^{k+1} e^{w(t-s)}\,ds$$
$$+ CM \int_{[t]}^{t} e^{w(t-s)}\,ds, \qquad t \geqslant 1.$$

This gives

$$\|x(t)\| \leqslant M_1 e^{wt} + M_2 e^{wt} \sum_{k=1}^{\infty} e^{-wk} + M_3 e^{wt}$$

and proves the assertion since the series converges, and any value of $t \geqslant 0$ can be placed in a segment $[n, n+1]$.

THEOREM 3.13. *If the operators $A_k \in L(E, E)$ and the function $\phi(t)$ is bounded on* $(-\infty, 0]$, *then solution* (3.25) *satisfies* (3.29).

4. EQUATIONS WITH VARIABLE COEFFICIENTS

Along with the equation

$$x'(t) = A(t)\, x(t) + B(t)\, x([t]), \qquad 0 \leqslant t < \infty, \tag{4.1}$$

we also consider

$$x'(t) = A(t)\, x(t). \tag{4.2}$$

If $A(t)$ is a bounded linear operator strongly continuous in t, the Cauchy problem

$$x(0) = c_0 \in E \tag{4.3}$$

for (4.2) has a unique solution. The value of the solution $x(t)$ to (4.2)–(4.3) at the moment t is denoted as

$$x(t) = U(t)\, c_0.$$

The operator $U(t)$ can be considered as the solution of the Cauchy problem

$$U'(t) = A(t)\, U(t), \qquad U(0) = I, \tag{4.4}$$

for the differential equation in the space of bounded operators acting in E. For each t there exists a bounded inverse operator $U^{-1}(t)$. The solution of the generalized Cauchy problem

$$x(s) = c_0, \qquad s \in [0, \infty)$$

for (4.2) is represented in the form

$$x(t) = U(t)\, U^{-1}(s)\, c_0 = U(t, s)\, c_0,$$

where $U(t, s)$ is the evolution operator.

THEOREM 4.1. *If $A(t)$, $B(t) \in L(E, E)$ and are strongly continuous on $0 \leqslant t < \infty$, then there exists a unique solution of problm* (4.1)–(4.3) *given by the formula*

$$x(t) = U(t) \left(U^{-1}([t]) + \int_{[t]}^{t} U^{-1}(s)\, B(s)\, ds \right) c_{[t]}, \tag{4.5}$$

where

$$c_{[t]} = \prod_{k=[t]}^{1} U(k) \left(U^{-1}(k-1) + \int_{k-1}^{k} U^{-1}(s)\, B(s)\, ds\right) c_0. \tag{4.6}$$

Proof. For $n \leqslant t < n+1$ we have the equation

$$x'(t) = A(t)\, x(t) + B(t)\, x(n).$$

Its solution $x_n(t)$ satisfying the condition $x(n) = c_n$ is given by the expression

$$x_n(t) = U_n(t) \left(I + \int_n^t U_n^{-1}(s)\, B(s)\, ds\right) c_n, \tag{4.7}$$

$U_n(t)$ being the solution of Eq. (4.4) such that $U(n) = I$. The solution of (4.1) for $t \in [n-1, n)$, with the condition $x(n-1) = c_{n-1}$, is

$$x_{n-1}(t) = U_{n-1}(t) \left(I + \int_{n-1}^{t} U_{n-1}^{-1}(s)\, B(s)\, ds\right) c_{n-1}.$$

From $x_{n-1}(n) = c_n$ we obtain the relation

$$c_n = U_{n-1}(n) \left(I + \int_{n-1}^{n} U_{n-1}^{-1}(s)\, B(s)\, ds\right) c_{n-1}.$$

To prove (4.6), it remains to observe that $U_i(t) = U(t)\, U^{-1}(i)$.

We can restate this theorem also in the case of unbounded operators.

THEOREM 4.2. *If the closed operators $A(t)$ and $B(t)$ are strongly continuous and have the common constant domain D for all $0 \leqslant t < \infty$ which is dense in E, the Cauchy problem for* (4.2) *is uniformly correct and $B(t): D \to D$, then* (4.5) *yields the unique solution of problem* (4.1)–(4.3) *with $c_0 \in D$.*

THEOREM 4.3. *If $A(t)$, $B(t) \in L(E, E)$ and are strongly continuous on $0 \leqslant t < \infty$, then solution* (4.5) *satisfies the estimate*

$$\|x(t)\| \leqslant e^{(t+1)a(t)} (b(t)+1)^{t+1} \|c_0\|, \tag{4.8}$$

where

$$a(t) = \max \|A(s)\|, \qquad b(t) = \max \|B(s)\|, \qquad 0 \leqslant s \leqslant t.$$

Proof. The operator $U_k(t) = U(t)\, U^{-1}(k)$ is the solution of Eq. (4.4) with the condition $U_k(k) = I$. Therefore, it is the sum of the series, convergent in the operator sense,

$$U_k(t) = I + \int_k^t A(s)\,ds + \int_k^t A(s)\,ds \int_k^s A(s_1)\,ds_1 + \cdots$$

whence

$$\| U_k(t) \| \leqslant e^{(t-k)a(t)},$$

and turning to (4.5) we find

$$\|x(t)\| \leqslant \left(e^{\{t\}a(t)} + b(t) \int_{[t]}^t e^{(t-s)a(t)}\,ds \right) \|c_{[t]}\|.$$

Hence,

$$\|x(t)\| \leqslant (b(t) + 1)\, e^{a(t)}\, \|c_{[t]}\|.$$

Since

$$\| U(k)\, U^{-1}(k-1) \| = \| U_{k-1}(k) \| \leqslant e^{a(t)},$$
$$\| U(k)\, U^{-1}(s) \| \leqslant e^{(k-s)a(t)},$$

we obtain from (4.6)

$$\|c_{[t]}\| \leqslant \prod_{k=1}^{[t]} \left(e^{a(t)} + b(t) \int_{k-1}^k e^{(k-s)a(t)}\,ds \right) \|c_0\|$$
$$\leqslant e^{[t]a(t)} (b(t) + 1)^{[t]}\, \|c_0\|.$$

Finally,

$$\|x(t)\| \leqslant e^{[t+1]a(t)} (b(t) + 1)^{[t+1]}\, \|c_0\|.$$

In particular, if the operators $A(t)$ and $B(t)$ are uniformly bounded on $[0, \infty)$, then

$$\|x(t)\| \leqslant M \exp(at + t \ln(b+1))$$

with some constant M, where $a = \max \|A(t)\|$, $b = \max \|B(t)\|$, $0 \leqslant t < \infty$. In this case, the conclusion about the solution growth is the same as for Eq. (3.11).

THEOREM 4.4. *Suppose the equation*

$$x'(t) = A(t)\,x(t) + f(t, x(t, x(t - [t]), x(t - 2[t]),..., x(t - N[t])) \tag{4.9}$$

satisfies the following hypotheses:

(i) *The operator* $A(t) \in L(E, E)$ *and is strongly continuous on* $R_+ = [0, \infty)$.

(ii) *The mapping* $f: R_+ \times E^N \to E$ *is continuous on the direct product of* R_+ *and* N *copies of* E.

(iii) *The function* $\phi: R_- \to E$ *is continuous on* $R_- = (-\infty, 0]$.

(iv) *The solution of the equation*

$$x_0'(t) = A(t)\,x_0(t) + f(t, x_0(t), x_0(t),..., x_0(t)), \qquad x_0(0) = \phi(0)$$

exists on $0 \leqslant t \leqslant 1$ *and is unique.*

Then the initial-value problem $x(t) = \phi(t)$, $t \in R_-$ *for* (4.9) *has a unique solution*

$$x(t) = U(t)\left(U^{-1}(1)\,x_0(1) + \sum_{k=1}^{[t-1]} \int_k^{k+1} U^{-1}(s) f_k(s)\,ds + \int_{[t]}^{t} U^{-1}(s) f_{[t]}(s)\,ds\right), \qquad t \geqslant 1 \tag{4.10}$$

where $U(t)$ *is the solution of* (4.4) *and*

$$f_k(s) = f(s, x_0(s-k), \phi(s-2k),..., \phi(s-Nk)).$$

Proof. On $n \leqslant t < n+1$, Eq. (4.9) assumes the form

$$x'(t) = A(t)\,x(t) + f_n(t).$$

Its solution $x_n(t)$ that satisfies the condition $x(n) = c_n$ is

$$x_n(t) = U_n(t)\left(c_n + \int_n^t U_n^{-1}(s) f_n(s)\,ds\right),$$

and $U_n(t) = U(t)\,U^{-1}(n)$. Hence

$$x_n(t) = U(t)\left(U^{-1}(n)\,c_n + \int_n^t U^{-1}(s) f_n(s)\,ds\right).$$

The same formula for $x_{n-1}(t)$ together with $x_{n-1}(n)=x_n(n)$ yields the result

$$c_n = U(n)\left(U^{-1}(n-1)\,c_{n-1} + \int_{n-1}^{n} U^{-1}(s) f_{n-1}(s)\,ds\right)$$

which leads to (4.10).

THEOREM 4.5. *The problem*

$$x'(t) = A(t)\,x(t) + \sum_{i=0}^{N} A_i(t)\,x([t-i]), \tag{4.11}$$

$$x(i) = c_i, \qquad i = 0, \ldots, N,$$

has a unique solution on $[N, \infty)$ *if* $A(t)$ *and all* $A_i(t) \in L(E, E)$ *and are strongly continuous on* $[0, \infty)$.

Proof. We consider Eq. (4.11) on the interval $n \leqslant t < n+1$ where $n \geqslant N$ is integer, with the initial conditions $x(n-i) = c_{n-i}$, $i = 0, \ldots, N$. Its solution $x_n(t)$ is given by

$$x_n(t) = U(t)\left(U^{-1}(n)\,c_n + \int_n^t U^{-1}(s)\left(\sum_{i=0}^{N} A_i(s)\,c_{n-i}\right) ds\right) \tag{4.12}$$

where $U(t)$ satisfies (4.4). The expression for $x_{n-1}(t)$ together with the condition $x_{n-1}(n) = x_n(n)$ leads to the equation

$$c_n = U(n)\left(U^{-1}(n-1)\,c_{n-1} + \int_{n-1}^{n} U^{-1}(s)\left(\sum_{i=0}^{N} A_i(s)\,c_{n-i-1}\right) ds\right) \tag{4.13}$$

which has a unique solution for the prescribed values of $c_0, \ldots, c_N$.

THEOREM 4.6. *The solution of the scalar equation*

$$x'(t) = ax(t) + \sum_{i=0}^{N} a_i(t)\,x([t-i]) \qquad (a = \text{const}), \tag{4.14}$$

$$x(i) = c_i, \qquad i = 0, \ldots, N,$$

tends to zero as $t \to \infty$ *if* $a < 0$, $a_i(t) \in C[N, \infty)$, *and* $\lim a_i(t) = 0$ *as* $t \to \infty$, *for* $0 \leqslant i \leqslant N$.

Proof. According to (4.12) and (4.13), the solution of (4.14) is given by

$$x(t) = e^{a\{t\}} c_{[t]} + \int_{[t]}^{t} e^{a(t-s)}\left(\sum_{i=0}^{N} a_i(s)\,c_{[t-i]}\right) ds, \tag{4.15}$$

where

$$c_{[t]} = \left(e^a + \int_{[t]}^{t} e^{a(t-s)} a_0(s)\, ds\right) c_{[t-1]}$$
$$+ \sum_{i=1}^{N} \left(\int_{[t-1]}^{[t]} e^{a([t]-s)} a_i(s)\, ds\right) c_{[t-i-1]}, \qquad t > N.$$

Take a number $\delta > 0$ such that

$$e^a + (N+1)\,\delta = q < 1.$$

There exists such a number T that, for $t > T$,

$$|a_i(t)| \leqslant \delta, \qquad i = 0, \ldots, N.$$

Then

$$|c_{[t]}| \leqslant (e^a + \delta)\, |c_{[t-1]}| + \delta \sum_{i=1}^{N} |c_{[t-i-1]}|. \tag{4.16}$$

To evaluate $c_{[t]}$, we employ the method developed in [14] for the study of distributional and analytic solutions to linear functional differential equations. Denote

$$M_{[t]} = \max |c_j|, \qquad 0 \leqslant j \leqslant [t].$$

From (4.16) it follows that

$$|c_{[t]}| \leqslant q M_{[t-1]}, \qquad t > T.$$

Hence, $M_{[t]} = M_{[t-1]}$, and starting with some k,

$$M_n = M_k, \qquad n \geqslant k. \tag{4.17}$$

The application of (4.17) to (4.16) successively yields

$$|c_{k+1}| \leqslant q M_k, \qquad |c_{k+2}| \leqslant q M_k, \ldots, |c_{k+N+1}| \leqslant q M_k.$$

Now we put $[t] = k + N + 2, \ldots, k + 2N + 2$ and use the latter inequalities to obtain

$$|c_{k+N+2}| \leqslant q^2 M_k, \qquad |c_{k+N+3}| \leqslant q^2 M_k, \ldots, |c_{k+2N+2}| \leqslant q^2 M_k.$$

Continuation of the iteration process shows that

$$|c_{k+(j-1)N+i}| \leqslant q^j M_k,$$

for all natural j and $i=j, j+1, \ldots, j+N$. This implies $c_{[t]} \to 0$ as $t \to \infty$, and the proof follows now from (4.15).

For the scalar problem

$$x'(t)=a(t)\,x(t)+a_0(t)\,x([t])+a_1(t)\,x([t-1]),$$
$$x(0)=c_0, \qquad x(1)=c_1$$

with coefficients continuous on $[0, \infty)$ we can indicate a simple algorithm of computing the solution. According to (4.12), we have

$$x_n(t)=u(t)\left(u^{-1}(n)+\int_n^t u^{-1}(s)\,a_0(s)\,ds\right)c_n$$
$$+\,u(t)\left(\int_n^t u^{-1}(s)\,a_1(s)\,ds\right)c_{n-1},$$

where

$$u(t)=\exp\left(\int_0^t a(s)\,ds\right).$$

Let $x_n(n)=c_n$ and

$$b_0(n)=u(n)\left(u^{-1}(n-1)+\int_{n-1}^n u^{-1}(s)\,a_0(s)\,ds\right),$$

$$b_1(n)=u(n)\int_{n-1}^n u^{-1}(s)\,a_1(s)\,ds.$$

Then

$$c_n=b_0(n)\,c_{n-1}+b_1(n)\,c_{n-2}, \qquad n \geqslant 2.$$

Denote

$$r_n=c_n/c_{n-1}.$$

From the relation

$$r_n=b_0(n)+\frac{b_1(n)}{r_{n-1}}$$

we obtain the continued-fraction expansion

$$r_n=b_0(n)+\frac{b_1(n)}{b_0(n-1)+}\,\frac{b_1(n-1)}{b_0(n-2)+}\cdots\frac{b_1(3)}{b_0(2)+}\,\frac{b_1(2)}{c_1/c_0},$$

and

$$c_n = r_1 r_2 \cdots r_n.$$

In conclusion, we consider the scalar equation

$$\begin{gathered} x'(t) = f(x(t), x([t]), x([t-1])), \\ x(0) = c_0, \qquad x(1) = c_1, \qquad 1 \leqslant t < \infty. \end{gathered} \tag{4.18}$$

If in the equation with parameters λ and μ,

$$x' = f(x, \lambda, \mu), \qquad f(x, \lambda, \mu) \in C(R^3), \tag{4.19}$$

$f(x, \lambda, \mu) \neq 0$ everywhere, then there exists the general integral

$$F(x, \lambda, \mu) = t + g(\lambda, \mu)$$

with an arbitrary function $g(\lambda, \mu)$. If $x(t)$ can be extended over all $t \geqslant 1$, then for the solution $x_n(t)$ of (4.18) on $n \leqslant t < n+1$ we have the equation

$$x_n' = f(x_n, c_n, c_{n-1}), \qquad x_n(n-1) = c_{n-1}, \qquad x_n(n) = c_n.$$

Hence, if $\lambda = c_n$, $\mu = c_{n-1}$, then

$$F(x_n, c_n, c_{n-1}) = t + g(c_n, c_{n-1}).$$

For $t = n$ this gives

$$F(c_n, c_n, c_{n-1}) = n + g(c_n, c_{n-1})$$

and

$$F(x_n, c_n, c_{n-1}) = t + F(c_n, c_n, c_{n-1}) - n. \tag{4.20}$$

Now we take the solution of (4.18) on $[n+1, n+2)$ that satisfies $x(n) = c_n$, $x(n+1) = c_{n+1}$. Since $x_n(n+1) = x_{n+1}(n+1)$, we put $t = n+1$ in (4.20) and get

$$F(c_{n+1}, c_n, c_{n-1}) = F(c_n, c_n, c_{n-1}) + 1.$$

This equation has a unique solution

$$c_{n+1} = h(c_n, c_{n-1}) \tag{4.21}$$

because

$$F(x, \lambda, \mu) = \int \frac{dx}{f(x, \lambda, \mu)}, \tag{4.22}$$

and $\partial F/\partial x \neq 0$. The difference equation (4.21) with the initial values c_0 and c_1 has a unique solution c_n for each $n \geqslant 2$ since the solution $x_n(t, c_n, c_{n-1})$ of (4.18) exists on $[n, n+1]$ and

$$c_2 = x_1(2, c_1, c_0), \qquad c_3 = x_2(3, c_2, c_1), \ldots,$$
$$c_{n+1} = x_n(n+1, c_n, c_{n-1}).$$

We state the following:

THEOREM 4.7. *If* $f(x, \lambda, \mu) \in C(R^3)$ *is different from zero everywhere and the solutions of* (4.19) *can be extended over* $[1, \infty)$, *then* (4.18) *has a unique solution given by formula* (4.20) *for* $t \in [n, n+1]$, *with* c_n *and* F *from* (4.21) *and* (4.22).

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